Theorem 6 $\quad U_{m\times n}$

U has orthonormal columns $\Leftrightarrow$ $U^TU = I$.

Proof.- Consider

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix}$$

U has orthonormal columns $\Leftrightarrow$ $\vec{u}_i \cdot \vec{u}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$

Also, $$U^TU = \begin{bmatrix} \vec{u}_1^T \rightarrow \\ \vec{u}_2^T \rightarrow \\ \vec{u}_n^T \rightarrow \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{bmatrix} =$$

$$= \begin{bmatrix} \vec{u}_1^T\vec{u}_1 & \vec{u}_1^T\vec{u}_2 & \cdots & \vec{u}_1^T\vec{u}_n \\ \vec{u}_2^T\vec{u}_1 & \vec{u}_2^T\vec{u}_2 & \cdots & \vec{u}_2^T\vec{u}_n \\ \vdots & & & \\ \vec{u}_n^T\vec{u}_1 & \cdots & \cdots & \vec{u}_n^T\vec{u}_n \end{bmatrix} = \begin{bmatrix} \vec{u}_1\cdot\vec{u}_1 & \vec{u}_1\cdot\vec{u}_2 & \cdots & \vec{u}_1\cdot\vec{u}_n \\ \vec{u}_2\cdot\vec{u}_1 & \vec{u}_2\cdot\vec{u}_2 & \cdots & \vec{u}_2\cdot\vec{u}_n \\ \vdots & & & \\ \vec{u}_n\cdot\vec{u}_1 & \vec{u}_n\cdot\vec{u}_2 & \cdots & \vec{u}_n\cdot\vec{u}_n \end{bmatrix}$$

Then

$U^TU = I \Leftrightarrow \vec{u}_i \cdot \vec{u}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \Leftrightarrow U$ has orthonormal columns.

Theorem 7. 1) $U_{m\times n}$ matrix with orthonormal columns.

2) $\vec{x}, \vec{y} \in \mathbb{R}^n$

Then

a) $\|U\vec{x}\| = \|\vec{x}\|$

b) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$

c) $(U\vec{x})\cdot(U\vec{y}) = 0 \iff \vec{x}\cdot\vec{y} = 0.$

Proof.

(b) $U\vec{x}\cdot U\vec{y} = (U\vec{y})^T U\vec{x} = (\vec{y}^T U^T)(U\vec{x})$

$= \vec{y}^T(U^T U\vec{x}) = \vec{y}^T(I\vec{x}) = \vec{y}^T\vec{x} = \vec{x}\cdot\vec{y}$

(a) $\|U\vec{x}\|^2 = U\vec{x}\cdot U\vec{x} \overset{(b)}{=} \vec{x}\cdot\vec{x} = \|\vec{x}\|^2$

$\iff \|U\vec{x}\| = \|\vec{x}\|.$

(c) $(U\vec{x})\cdot(U\vec{y}) = 0 \overset{(b)}{\iff} \vec{x}\cdot\vec{y} = 0$

<u>Theorem 10</u> $B = \{\vec{u}_1, .., \vec{u}_p\}$ is <u>an orthonormal basis</u> for $W \subset \mathbb{R}^n$.

Then

for $\vec{y} \in \mathbb{R}^n$,

a) $$\boxed{proj_W \vec{y} = (\vec{y}.\vec{u}_1)\vec{u}_1 + (\vec{y}.\vec{u}_2)\vec{u}_2 + .. + (\vec{y}.\vec{u}_p)\vec{u}_p} \quad (3.1)$$

b) If $U = [\vec{u}_1\ \vec{u}_2 \dots \vec{u}_p]$, then

$$proj_W \vec{y} = U U^T \vec{y}.$$

<u>Proof</u>:-

a) By definition

$$proj_W \vec{y} = \frac{\vec{y}.\vec{u}_1}{\vec{u}_1.\vec{u}_1}\vec{u}_1 + \dots + \frac{\vec{y}.\vec{u}_p}{\vec{u}_p.\vec{u}_p}\vec{u}_p$$

If B is an orthonormal basis $\Rightarrow$ $\vec{u}_i . u_i = 1$ and (3.1) follows.

b)

$$U^T \vec{y} = \begin{bmatrix} \vec{u}_1^T \rightarrow \\ \vdots \\ \vec{u}_p^T \rightarrow \end{bmatrix} \begin{bmatrix} \vec{y} \\ \downarrow \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T \vec{y} \\ \vdots \\ \vec{u}_p^T \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{u}_1 \cdot \vec{y} \\ \vdots \\ \vec{u}_p \cdot \vec{y} \end{bmatrix}$$

$$\Rightarrow U(U^T \vec{y}) = \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_p \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \vec{u}_1 \cdot \vec{y} \\ \vdots \\ \vec{u}_p \cdot \vec{y} \end{bmatrix} =$$

$$= (\vec{u}_1 \cdot \vec{y})\vec{u}_1 + \dots (\vec{u}_p \cdot \vec{y})\vec{u}_p \overset{\text{Same as (3.1)}}{=\!=\!=} proj_W \vec{y}.$$